ISSN 0012-2661, Differential Equations, 2009, Vol. 45, No. 1, pp. 33–45. © Pleiades Publishing, Ltd., 2009.
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ORDINARY DIFFERENTIAL EQUATIONS

On the Relationship Between Solutions of Delay Differential Equations and Infinite-Dimensional Systems of Differential Equations

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Received March 14, 2007

Abstract—We study the limit properties of solutions for a class of systems of ordinary differential equations as the number of equations and a certain parameter grow unboundedly. We show that the sequence of functions formed by the last components of solutions of such systems has a repeated limit. The limit function is a solution of a delay differential equation. Estimates of the convergence rate are obtained.

DOI: 10.1134/S0012266109010042

1. INTRODUCTION

The theory of delay differential equations was intensively developing in the second half of the 20th century. This was related to numerous applied problems whose analysis necessitated solving delay equations. Equations of this type arise in the description of processes whose rate is determined by their previous states. Such processes are often referred to as "delay processes" or "processes with aftereffect." A reasonably complete introduction to the theory of delay differential equations can be found, e.g., in [1–3].

The present paper deals with yet another problem that leads to the study of delay differential equations. More precisely, we continue the analysis of relationships, established in [4], between solutions of a class of systems of ordinary differential equations of infinite dimension and solutions of equations of the form

$$\frac{d}{dt}y(t) = f(y(t), qy(t-\tau)), \qquad t > \tau. \tag{1.1}$$

The Cauchy problem for the following system of ordinary differential equations was considered in [4] :

$$\begin{gathered}\frac{dx}{dt} = A_n(\tau,\theta)x + h(q\theta y_n), \qquad \frac{dy_n}{dt} = f(y_n, x_n),\\ x_1|_{t=0} = \cdots = x_n|_{t=0} = 0, \qquad y_n|_{t=0} = y_0,\end{gathered} \tag{1.2}$$

where $A_n(\tau,\theta)$ is the bidiagonal $n\times n$ matrix with main diagonal

$$(-(n-1)/\tau, \ldots, -(n-1)/\tau, -\theta)$$

and with subdiagonal entries $(n-1)/\tau$, $h(\xi)$ is the column vector of the form

$$h(\xi) = \mathrm{col}(\xi, 0, \ldots, 0), \qquad \xi = q\theta y_n, \qquad \tau, q > 0$$

are fixed parameters, and $\theta > 0$; moreover, $f(u,v)$ is a bounded function satisfying the Lipschitz condition with respect to both variables:

$$\sup_{u,v\in R} |f(u,v)| = F < \infty, \qquad |f(u_1,v_1) - f(u_2,v_2)| \le L_1|u_1-u_2| + L_2|v_1-v_2|. \tag{1.3}$$

It was proved in [4] that some class of solutions of Eq. (1.1) can be represented as the repeated limit

$$\lim_{\theta\to\infty}\lim_{n\to\infty} y_n(t,\theta) = y(t), \tag{1.4}$$

where $y_n(t,\theta)$ is the last component of the solution of the Cauchy problem (1.2). The limit relation (1.4) was proved for $q \in (0,1)$ on the interval $(\tau, T_0]$, where

$$\tau < T_0 < \min\left\{\frac{1-q}{L_1}, \frac{1}{L_2}\right\}.$$

In the present paper, we strengthen this result by proving the limit relation (1.4) for any parameter $q > 0$ on an arbitrary interval $(\tau, T]$ and by estimating the convergence rate.

Note that relationships of this type between solutions of systems of ordinary differential equations of infinite size and solutions of delay differential equations were obtained in [5] when modeling unbranched multistage synthesis of a material [6]. The corresponding system of differential equations has the form

$$\begin{aligned} \frac{dx_1}{dt} &= g(x_n) - \frac{n-1}{\tau}x_1, \\ \frac{dx_i}{dt} &= \frac{n-1}{\tau}(x_{i-1} - x_i), \qquad i = 2,\dots,n-1, \\ \frac{dx_n}{dt} &= \frac{n-1}{\tau}x_{n-1} - \theta x_n. \end{aligned} \tag{1.5}$$

It was shown in [5] that if the number n of equations in system (1.5) tends to infinity and only the last components of the solution of the Cauchy problem with zero initial data $x|_{t=0} = 0$ are considered, then we obtain a uniformly convergent sequence

$$x_n(t) \to y(t), \qquad n\to\infty, \qquad t \in [0,T];$$

moreover, the limit function $y(t)$ satisfies the identity

$$\frac{dy(t)}{dt} \equiv -\theta y(t) + g(y(t-\tau)), \qquad t > \tau.$$

Consequently, the function $y(t)$ is a solution of the delay differential equation (1.1) with right-hand side

$$f(u,v) = -\theta u + g(v).$$

These results were generalized in [7] to a wide class of quasilinear differential equations.

2. SYSTEMS WITH INFINITELY MANY DIFFERENTIAL EQUATIONS

The last two components of the solution of problem (1.2) are solutions of the following system of integral equations:

$$x_n(t,\theta) = q\theta\int_0^t \psi_n(t-s,\theta)y_n(s,\theta)\,ds, \tag{2.1}$$

$$y_n(t,\theta) = y_0 + \int_0^t f(y_n(s,\theta), x_n(s,\theta))\,ds, \tag{2.2}$$

where

$$\psi_n(t,\theta) = \frac{e^{-\theta t}}{(1-\theta\tau/(n-1))^{n-1}}S_n(t,\theta),$$

$$S_n(t,\theta) = 1 - e^{-\omega t}\sum_{k=0}^{n-2}\frac{(\omega t)^k}{k!}, \qquad \omega = \frac{n-1}{\tau} - \theta.$$

Indeed, Eq. (2.1) can be obtained by taking into account the initial conditions $x|_{t=0} = 0$ and by applying the Cauchy formula to the first n equations in (1.2), and Eq. (2.2) can be obtained just by integrating the last equation in (1.2).

Now consider the sequence of Cauchy problems (1.2) obtained as the number n of equations tends to infinity. By solving each of the problems and by considering only the last two components of the solutions, we obtain the sequence $\{z_n(t,\theta)\}$ of vector functions

$$z_n(t,\theta) = (x_n(t,\theta), y_n(t,\theta)).$$

The following lemma implies that, for fixed θ, this sequence is convergent in the space $C[0,T] \times C[0,T]$ of continuous vector functions.

Theorem 2.1. *Let $x_n(t,\theta)$ and $y_n(t,\theta)$ satisfy the system of integral equations* (2.1), (2.2). *Then the sequence $\{x_n(t,\theta),\ y_n(t,\theta)\}$ is uniformly convergent on any closed interval* $[0,T]$ *for any $\theta > 0$ as $n \to \infty$.*

Proof. It suffices to prove that the sequence is Cauchy; together with the completeness of the space $C[0,T] \times C[0,T]$, this implies the desired convergence.

Let us estimate the absolute values of the differences $|x_n(t,\theta) - x_{n+l}(t,\theta)|$ and $|y_n(t,\theta) - y_{n+l}(t,\theta)|$, $l \in N$. Since the function $f(u,v)$ is bounded, it follows from Eq. (2.2) that the sequence $\{y_n(t,\theta)\}$ is uniformly bounded for any n and on any closed interval $[0,T]$:

$$\max_{t\in[0,T]} |y_n(t,\theta)| \le Y = |y_0| + TF.$$

After simple transformations, we obtain the inequalities

$$\begin{aligned}|x_n(t,\theta) - x_{n+l}(t,\theta)| &\le q\theta \int_0^t |\psi_{n+l}(t-s,\theta) - \psi_n(t-s,\theta)|\,|y_{n+l}(s,\theta)|\,ds \\ &\quad + q\theta \int_0^t |\psi_n(t-s,\theta)|\,|y_{n+l}(s,\theta) - y_n(s,\theta)|\,ds \\ &\le I_{n,l}(t,\theta) + q\theta\Psi_n \int_0^t |y_{n+l}(s,\theta) - y_n(s,\theta)|\,ds,\end{aligned}$$

$$|y_n(t,\theta) - y_{n+l}(t,\theta)| \le L_1 \int_0^t |y_n(s,\theta) - y_{n+l}(s,\theta)|\,ds + L_2 \int_0^t |x_n(s,\theta) - x_{n+l}(s,\theta)|\,ds,$$

where

$$I_{n,l}(t,\theta) = q\theta \int_0^t |\psi_{n+l}(t-s,\theta) - \psi_n(t-s,\theta)|\,|y_{n+l}(s,\theta)|\,ds, \qquad \Psi_n(\theta) = \max_{\xi\in[0,T]} \psi_n(\xi,\theta). \tag{2.3}$$

By the definition of the function $\psi_n(\xi,\theta)$, we have the inequality

$$\Psi_n(\theta) = \max_{\xi\in[0,T]} |\psi_n(\xi,\theta)| \le \frac{1}{(1-\theta\tau/(n-1))^{n-1}}.$$

Since $\lim_{n\to\infty}(1-\theta\tau/(n-1))^{-n+1} = e^{\theta\tau}$, it follows that there exists an n_0 such that

$$\Psi_n(\theta) \le 2e^{\theta\tau} = \Psi(\theta)$$

for any $n \ge n_0$.

By using the notation

$$x_{n,l}(t,\theta) = x_{n+l}(t,\theta) - x_n(t,\theta), \qquad y_{n,l}(t,\theta) = y_{n+l}(t,\theta) - y_n(t,\theta),$$
$$M_{n,l}(\theta) = \max_{t\in[0,T]} I_{n,l}(t,\theta),$$

we arrive at the system of integral inequalities

$$|x_{n,l}(t,\theta)| \le M_{n,l}(\theta) + q\theta\Psi(\theta)\int_0^t |y_{n,l}(s,\theta)|\,ds, \tag{2.4}$$

$$|y_{n,l}(t,\theta)| \le L_1\int_0^t |y_{n,l}(s,\theta)|\,ds + L_2\int_0^t |x_{n,l}(s,\theta)|\,ds. \tag{2.5}$$

For given $l \in N$, we denote the right-hand sides of inequalities (2.4) and (2.5) by $U_{n,l}(t,\theta)$ and $V_{n,l}(t,\theta)$, respectively. By differentiating $U_{n,l}$ and $V_{n,l}$ with respect to t, we obtain

$$\frac{dU_{n,l}(t,\theta)}{dt} = q\theta\Psi(\theta)|y_{n,l}(t,\theta)| \le q\theta\Psi(\theta)V_{n,l}(t,\theta), \tag{2.6}$$
$$\frac{dV_{n,l}(t,\theta)}{dt} = L_1|y_{n,l}(t,\theta)| + L_2|x_{n,l}(t,\theta)| \le L_1V_{n,l}(t,\theta) + L_2U_{n,l}(t,\theta); \tag{2.7}$$

in addition, $U_{n,l}(0,\theta) = M_{n,l}(\theta)$ and $V_{n,l}(0,\theta) = 0$.

It follows from (2.7) that

$$V_{n,l}(t,\theta) \le L_2\int_0^t e^{L_1(t-s)}U_{n,l}(s,\theta)\,ds. \tag{2.8}$$

Then, by taking into account (2.6), we obtain

$$U_{n,l}(t,\theta) \le M_{n,l}(\theta) + q\theta\Psi(\theta)L_2\int_0^t\int_0^\xi e^{L_1(\xi-s)}U_{n,l}(s,\theta)\,ds\,d\xi. \tag{2.9}$$

By denoting the right-hand side of inequality (2.9) by $W_{n,l}(t,\theta)$ and by following the same scheme, we obtain the differential inequality

$$W''_{n,l} - \alpha L_1W'_{n,l} - \alpha W_{n,l} \le 0, \qquad W_{n,l}(0,\theta) = M_{n,l}(\theta), \qquad W'_{n,l}(0,\theta) = 0, \tag{2.10}$$

where $\alpha = q\theta\Psi(\theta)L_2$.

We rewrite the differential inequality (2.10) in the form

$$\left(\frac{d}{dt} - \lambda_1 I\right)\left(\frac{d}{dt} - \lambda_2 I\right)W_{n,l} \le 0,$$

where

$$\lambda_1 = (\alpha L_1 + \sqrt{\alpha^2L_1^2 + 4\alpha}\,)/2, \qquad \lambda_2 = (\alpha L_1 - \sqrt{\alpha^2L_1^2 + 4\alpha}\,)/2 \tag{2.11}$$

are the roots of the equation $\lambda^2 - \alpha L_1\lambda - \alpha = 0$.

Now, by integrating this inequality and by taking into account the initial data, we obtain

$$W_{n,l}(t,\theta) \le \frac{M_{n,l}(\theta)}{\lambda_1 - \lambda_2} = \frac{1}{\lambda_1 - \lambda_2}\max_{s\in[0,T]} I_{n,l}(s,\theta)(\lambda_1e^{\lambda_2t} - \lambda_2e^{\lambda_1t}). \tag{2.12}$$

Note that [4]

$$\max_{t\in[0,T]} I_{n,l}(t,\theta) \to 0, \qquad n\to\infty. \tag{2.13}$$

Since, by (2.4) and (2.9),

$$|x_{n,l}(t,\theta)| \le U_{n,l}(t,\theta) \le W_{n,l}(t,\theta), \tag{2.14}$$

it follows that $\{x_n(t,\theta)\}$ is a Cauchy sequence.

Now let us show that $\{y_n(t,\theta)\}$ is a Cauchy sequence on any interval $[0,T]$. By virtue of inequalities (2.5), (2.8), and (2.14), we have

$$|y_{n,l}(t,\theta)| \le V_{n,l}(t,\theta) \le L_2 \int\limits_0^t e^{L_1(t-s)} U_{n,l}(s,\theta)\,ds \le L_2 \int\limits_0^t e^{L_1(t-s)} W_{n,l}(s,\theta)\,ds. \tag{2.15}$$

Consequently, by taking into account the estimate (2.12) and the convergence (2.13), we obtain the relation

$$\max_{t\in[0,T]} |y_{n,l}(t,\theta)| \to 0, \qquad n\to\infty,$$

on any closed interval $[0,T]$; i.e., $\{y_n(t,\theta)\}$ is a Cauchy sequence on $[0,T]$. Since the space $C[0,T]\times C[0,T]$ is complete, it follows that there exists a vector function

$$z(t,\theta) = \lim_{n\to\infty} z_n(t,\theta) \in C[0,T]\times C[0,T].$$

The proof of the theorem is complete.

Corollary. *The limit vector function $z(t,\theta) = (x(t,\theta), y(t,\theta))$ is a solution of the system of integral equations*

$$x(t,\theta) = q\theta \int\limits_0^{t-\tau} e^{-\theta(t-\tau-s)} y(s,\theta)\,ds, \qquad t>\tau, \tag{2.16}$$

$$y(t,\theta) = y_0 + \int\limits_0^t f(y(s,\theta), x(s,\theta))\,ds \tag{2.17}$$

with $x(t,\theta)=0$, $t\in[0,\tau]$.

Theorem 2.2. *The system of integral equations* (2.16), (2.17) *has a unique continuous solution on* $[0,T]$ *such that* $x(t,\theta)=0$, $t\in[0,\tau]$.

Proof. The existence of a solution has been proved above. Let us prove the uniqueness.

Suppose the contrary: there exist two distinct solutions $(x_1(t,\theta), y_1(t,\theta))$ and $(x_2(t,\theta),\ y_2(t,\theta))$ satisfying system (2.16), (2.17); moreover, $x_1(t,\theta)=x_2(t,\theta)=0$, $t\in[0,\tau]$. Consider the difference

$$x(t,\theta) = x_1(t,\theta) - x_2(t,\theta), \qquad y(t,\theta) = y_1(t,\theta) - y_2(t,\theta)$$

of these solutions. Then $x(t,\theta)$ satisfies Eq. (2.16) and $y(t,\theta)$ satisfies the equation

$$y(t,\theta) = \int\limits_0^t [f(y_1(s,\theta), x_1(s,\theta)) - f(y_2(s,\theta), x_2(s,\theta))]\,ds, \qquad t>0. \tag{2.18}$$

Hence we have

$$|x(t,\theta)| \le q\theta \int\limits_0^{t-\tau} |y(s,\theta)|\,ds, \qquad t>\tau, \tag{2.19}$$

$$x(t,\theta)\equiv 0, \quad 0\le t\le\tau, \qquad |y(t,\theta)| \le \int\limits_0^t [L_1|y(s,\theta)| + L_2|x(s,\theta)|]\,ds. \tag{2.20}$$

Then on the interval $[0, \tau]$, we obtain the inequality

$$|y(t, \theta)| \le \int_0^t L_1 |y(s, \theta)|\, ds, \tag{2.21}$$

and since $y(0, \theta) = 0$, it follows from the Gronwall inequality that

$$y(t, \theta) \equiv 0, \quad t \in [0, \tau].$$

This, together with inequality (2.19), implies that $x(t, \theta) \equiv 0$ for $\tau \le t \le 2\tau$.

Returning to inequality (2.20), we obtain the estimate (2.21) on $[\tau, 2\tau]$, which, just as above, implies that $y(t, \theta) \equiv 0$ for $\tau \le t \le 2\tau$. By repeating these considerations, we find that system (2.16), (2.18) has only the zero solution on any interval $[0, T]$. The resulting contradiction implies the uniqueness of the solution of the system of integral equations (2.16), (2.17). The proof of the theorem is complete.

3. PROPERTIES OF SOLUTIONS OF THE INTEGRAL SYSTEM AS $\theta \to \infty$

Let us study the behavior of the solution $z(t, \theta)$ of system (2.16), (2.17) as $\theta \to \infty$. For simplicity, we consider the case in which θ is a positive integer. By solving this system for various $\theta = m \in N$, we obtain the sequence $\{z^m(t)\}$ of vector functions $z^m(t) = (x^m(t), y^m(t))$, where

$$x^m(t) \equiv qm \int_0^{t-\tau} e^{-m(t-\tau-s)} y^m(s)\, ds, \qquad t > \tau, \tag{3.1}$$

$$y^m(t) \equiv y_0 + \int_0^t f(y^m(s), x^m(s))\, ds, \tag{3.2}$$

and $x^m(t) \equiv 0$ for $t \in [0, \tau]$.

Theorem 3.1. *Let the function $f(u, v)$ satisfy condition* (1.3). *Then the sequence $\{z^m(t)\}$ is convergent on $(\tau, T]$ for any $T > \tau$:*

$$x^m(t) \to \mathcal{X}(t), \qquad m \to \infty, \tag{3.3}$$

$$y^m(t) \to \mathcal{Y}(t), \qquad m \to \infty; \tag{3.4}$$

moreover,

$$\mathcal{X}(t) = q\mathcal{Y}(t - \tau),$$

$$\mathcal{Y}(t) = y_0 + \int_0^t f(\mathcal{Y}(s), 0)\, ds, \qquad t \in [0, \tau], \tag{3.5}$$

$$\mathcal{Y}(t) = y_0 + \int_0^\tau f(\mathcal{Y}(s), 0)\, ds + \int_\tau^t f(\mathcal{Y}(s), q\mathcal{Y}(s - \tau))\, ds, \qquad t \in (\tau, T]. \tag{3.6}$$

Proof. Just as in [4], we prove the convergence (3.3), (3.4) successively on the intervals $(j\tau, (j + 1)\tau] \subset (\tau, T]$.

First, consider the interval $[0, \tau]$. Since $x^m(t) \equiv 0$, it follows from (3.2) that

$$y^m(t) \equiv y_0 + \int_0^t f(y^m(s), 0)\, ds;$$

i.e., the sequence $\{y^m(t)\}$ is stationary, $y^m(t) \equiv y(t)$, $m \in N$. By definition, we set $\mathcal{Y}(t) = y(t)$, $t \in [0, \tau]$. Consequently, the function $\mathcal{Y}(t)$ is a solution of Eq. (3.5) for $t \in [0, \tau]$.

Consider the interval $(\tau, 2\tau]$. By (3.1),

$$x^m(t) = qm \int_0^{t-\tau} e^{-m(t-\tau-s)} \mathcal{Y}(s)\, ds.$$

It was shown in [4] that the sequence $\{x^m(t)\}$ is uniformly convergent on $[\tau+\delta, 2\tau]$ for any $\delta \in (0, \tau)$,

$$x^m(t) \to \mathcal{X}(t) \equiv q\mathcal{Y}(t-\tau), \qquad m \to \infty;$$

in addition, one has the estimate

$$|x^m(t) - q\mathcal{Y}(t-\tau)| \le \frac{qF}{m} + q(|y_0| + \tau F)e^{-\delta m}. \tag{3.7}$$

Let us show that the sequence $\{y^m(t)\}$ is also uniformly convergent on $[\tau, 2\tau]$:

$$y^m(t) \to \mathcal{Y}(t), \qquad m \to \infty.$$

By (3.2) and the condition imposed on the function $f(u, v)$, we obtain

$$\begin{aligned} |y^{m+l}(t) - y^m(t)| &\le \int_\tau^{\tau+\delta} |f(y^{m+l}(s), x^{m+l}(s)) - f(y^m(s), x^m(s))|\, ds \\ &\quad + \int_{\tau+\delta}^{t} |f(y^{m+l}(s), x^{m+l}(s)) - f(y^m(s), x^m(s))|\, ds \\ &\le 2\delta F + L_1 \int_\tau^t |y^{m+l}(s) - y^m(s)|\, ds + \tau L_2 \max_{[\tau+\delta, 2\tau]} |x^{m+l}(t) - x^m(t)|. \end{aligned}$$

This, together with inequality (3.7), implies that

$$|y^{m+l}(t) - y^m(t)| \le 2\delta F + 2\tau L_2 \left(\frac{qF}{m} + q(|y_0| + \tau F)e^{-m\delta} \right) + L_1 \int_\tau^t |y^{m+l}(s) - y^m(s)|\, ds.$$

By using the Gronwall inequality, on the interval $[\tau, 2\tau]$, we obtain the estimate

$$|y^{m+l}(t) - y^m(t)| \le 2 \left(\delta F + \tau L_2 \left(\frac{qF}{m} + q(|y_0| + \tau F)e^{-\delta m} \right) \right) e^{L_1 t}. \tag{3.8}$$

Since $\delta > 0$ is arbitrary, it follows that the sequence $\{y^m(t)\}$ is Cauchy in $C[\tau, 2\tau]$. Therefore, the convergence (3.4) holds.

By virtue of (3.3) and (3.4), in (3.2), one can pass to the limit on the interval $[\tau, 2\tau]$ as $m \to \infty$ and obtain (3.6). Indeed, by (3.2) and condition (1.3), we have

$$\begin{aligned}
&\left| y^m(t) - \left(y_0 + \int_0^\tau f(\mathcal{Y}(s), 0)\, ds + \int_\tau^t f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds \right) \right| \\
&\quad = \left| \int_\tau^t [f(y^m(s), x^m(s)) - f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\quad \le \left| \int_\tau^{\tau+\delta} [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\qquad + \left| \int_{\tau+\delta}^t [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\qquad + \left| \int_\tau^t [f(y^m(s), q\mathcal{Y}(s-\tau)) - f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\quad \le 2\delta F + \tau L_2 \max_{[\tau+\delta, 2\tau]} |x^m(t) - q\mathcal{Y}(t-\tau)| + \tau L_1 \max_{[\tau, 2\tau]} |y^m(t) - \mathcal{Y}(t)|.
\end{aligned}$$

This, together with (3.3) and (3.4), implies the inequality

$$\overline{\lim_{m\to\infty}} \left| y^m(t) - \left(y_0 + \int_0^\tau f(\mathcal{Y}(s), 0)\, ds + \int_\tau^t f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds \right) \right| \le 2\delta F.$$

Therefore, since $\delta > 0$ is arbitrary, we have

$$y^m(t) \to y_0 + \int_0^\tau f(\mathcal{Y}(s), 0)\, ds + \int_\tau^t f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds, \qquad m \to \infty,$$

for $t \in [\tau, 2\tau]$. On the other hand, $y^m(t) \to \mathcal{Y}(t)$ as $m \to \infty$. Therefore, relation (3.6) on the interval $(\tau, 2\tau]$ follows from the uniqueness of the limit.

By letting l in inequality (3.8) tend to infinity, we obtain the estimate

$$\max_{[\tau,2\tau]} |y^m(t) - \mathcal{Y}(t)| \le 2 \left(\delta F + \tau L_2 \left(\frac{qF}{m} + q(|y_0| + \tau F) e^{-\delta m} \right) \right) e^{2\tau L_1}. \tag{3.9}$$

Note that relations (3.2) and (3.6) imply the estimates

$$\max_{[0,t]} |y^m(s)| \le |y_0| + tF, \qquad \max_{[0,t]} |\mathcal{Y}(s)| \le |y_0| + tF. \tag{3.10}$$

Let us show that, by using the resulting estimates, one can prove relations (3.3) and (3.4) on the interval $(2\tau, 3\tau]$. We represent $x^m(t)$ as the sum of two terms,

$$\begin{aligned}
x^m(t) &= qm \int_0^{t-\tau} e^{-m(t-\tau-s)} y^m(s)\, ds \\
&= qm \int_0^\tau e^{-m(t-\tau-s)} y^m(s)\, ds + qm \int_\tau^{t-\tau} e^{-m(t-\tau-s)} y^m(s)\, ds \\
&= X_1^m(t) + X_2^m(t).
\end{aligned} \tag{3.11}$$

By taking into account (3.10), for the first term, we obtain the estimate

$$|X_1^m(t)| \le q(|y_0| + \tau F)m \int\limits_0^\tau e^{-m(t-\tau-s)}\, ds.$$

Then

$$\max_{[2\tau+\delta,3\tau]} |X_1^m(t)| \le q(|y_0| + \tau F)e^{-\delta m} \tag{3.12}$$

for any $\delta \in (0, \tau)$. It was proved in [4] that

$$X_2^m(t) \to q\mathcal{Y}(t-\tau) \equiv \mathcal{X}(t), \qquad m \to \infty. \tag{3.13}$$

Let us present some considerations in [4], which are used in the proof of (3.13) and are needed in Section 4.

Set

$$U^m(t) = X_2^m(t) - q\mathcal{Y}(t-\tau) = qm \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)} y^m(s)\, ds - q\mathcal{Y}(t-\tau). \tag{3.14}$$

By taking into account the relation

$$1 - e^{-m(t-2\tau)} = m \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)}\, ds,$$

we represent $U^m(t)$ in the form

$$U^m(t) = qm \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)}(y^m(s) - \mathcal{Y}(s))\, ds$$
$$+ qm \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)}(\mathcal{Y}(s) - \mathcal{Y}(t-\tau))\, ds - q\mathcal{Y}(t-\tau)e^{-m(t-2\tau)}.$$

Then, by (3.6), we have

$$U^m(t) = qm \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)}(y^m(s) - \mathcal{Y}(s))\, ds$$
$$+ qm \int\limits_\tau^{t-\tau} e^{-m(t-\tau-s)} \left[\int\limits_{t-\tau}^{s} f(\mathcal{Y}(\xi), q\mathcal{Y}(\xi-\tau))\, d\xi \right] ds - q\mathcal{Y}(t-\tau)e^{-m(t-2\tau)}$$
$$= U_1^m(t) + U_2^m(t) + U_3^m(t). \tag{3.15}$$

By taking into account the convergence (3.4), we have the uniform convergence $U_1^m(t) \to 0$, $m \to \infty$. Since the function $f(u, v)$ is bounded, we readily obtain the estimate

$$|U_2^m(t)| \le \frac{qF}{m} \int\limits_0^\infty e^{-\eta}\eta\, d\eta = \frac{qF}{m}, \tag{3.16}$$

and by using the second inequality in (3.10), we arrive at the estimate

$$|U_3^m(t)| \le q(|y_0| + 2\tau F)e^{-\delta m}. \tag{3.17}$$

Consequently, for any $t \in (2\tau, 3\tau]$, we have the convergence $U^m(t) \to 0$, $m \to \infty$; moreover, it is uniform on $[2\tau + \delta, 3\tau]$ for any $\delta \in (0, \tau)$. This readily implies the convergence (3.13). From (3.13) and the estimate (3.12), we have the convergence (3.3) on the interval $[2\tau + \delta, 3\tau]$ for any $\delta \in (0, \tau)$.

Now let us consider the sequence $\{y^m(t)\}$ and show that it is uniformly convergent on the interval $[2\tau, 3\tau]$.

We represent (3.2) in the form

$$\begin{aligned} y^m(t) &= y_0 + \int_0^\tau f(y^m(s), 0)\, ds + \int_\tau^{2\tau} f(y^m(s), x^m(s))\, ds + \int_{2\tau}^t f(y^m(s), x^m(s))\, ds \\ &= y^m(2\tau) + \int_{2\tau}^t [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \\ &\quad + \int_{2\tau}^t f(y^m(s), q\mathcal{Y}(s-\tau))\, ds. \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned} y^{m+l}(t) - y^m(t) &= y^{m+l}(2\tau) - y^m(2\tau) + \int_{2\tau}^{2\tau+\delta} [f(y^{m+l}(s), x^{m+l}(s)) - f(y^{m+l}(s), q\mathcal{Y}(s-\tau))]\, ds \\ &\quad + \int_{2\tau+\delta}^t [f(y^{m+l}(s), x^{m+l}(s)) - f(y^{m+l}(s), q\mathcal{Y}(s-\tau))]\, ds \\ &\quad - \int_{2\tau}^{2\tau+\delta} [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \\ &\quad - \int_{2\tau+\delta}^t [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \\ &\quad + \int_{2\tau}^t [f(y^{m+l}(s), q\mathcal{Y}(s-\tau)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \end{aligned}$$

for any $l \geq 1$. Since the function $f(u, v)$ is bounded, we have the inequality

$$\begin{aligned} |y^{m+l}(t) - y^m(t)| &\leq 4\delta F + |y^{m+l}(2\tau) - y^m(2\tau)| + \tau L_2 \max_{[2\tau+\delta, 3\tau]} |x^m(s) - q\mathcal{Y}(s-\tau)| \\ &\quad + \tau L_2 \max_{[2\tau+\delta, 3\tau]} |x^{m+l}(s) - q\mathcal{Y}(s-\tau)| + L_1 \int_{2\tau}^t |y^{m+l}(s) - y^m(s)|\, ds, \end{aligned}$$

which, in view of the Gronwall type inequality, acquires the form

$$\begin{aligned} |y^{m+l}(t) - y^m(t)| &\leq (4\delta F + |y^{m+l}(2\tau) - y^m(2\tau)| + \tau L_2 \max_{[2\tau+\delta, 3\tau]} |x^m(s) - q\mathcal{Y}(s-\tau)| \\ &\quad + \tau L_2 \max_{[2\tau+\delta, 3\tau]} |x^{m+l}(s) - q\mathcal{Y}(s-\tau)|) e^{L_1 t}, \quad t \in [2\tau, 3\tau]. \end{aligned} \tag{3.19}$$

Since $\delta > 0$ is arbitrary, it follows from the convergence (3.4) on $[\tau, 2\tau]$ and the uniform convergence of the sequence $\{x^m(t)\}$ on $[2\tau+\delta, 3\tau]$ that $\{y^m(t)\}$ is a Cauchy sequence in $C[2\tau, 3\tau]$. Consequently, the uniform convergence (3.4) occurs on $[2\tau, 3\tau]$.

It follows from the preceding considerations that the sequence $\{y^m(t)\}$ uniformly converges to the function $\mathcal{Y}(t)$ on the entire interval $[\tau, 3\tau]$.

Now, in view of (3.3) and (3.4), one can readily pass to the limit as $m \to \infty$ on the interval $[2\tau, 3\tau]$ and obtain relation (3.6). Indeed, by taking into account (3.18), we obtain

$$\begin{aligned}
&\left| y^m(t) - \left(\mathcal{Y}(2\tau) + \int_{2\tau}^{t} f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds \right) \right| \\
&\qquad \le |y^m(2\tau) - \mathcal{Y}(2\tau)| + \left| \int_{2\tau}^{2\tau+\delta} [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\qquad + \left| \int_{2\tau+\delta}^{t} [f(y^m(s), x^m(s)) - f(y^m(s), q\mathcal{Y}(s-\tau))]\, ds \right| \\
&\qquad + \left| \int_{2\tau}^{t} [f(y^m(s), q\mathcal{Y}(s-\tau)) - f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))]\, ds \right|.
\end{aligned}$$

This, together with the uniform convergence (3.3) on the interval $[2\tau + \delta, 3\tau]$ and the uniform convergence (3.4) on the interval $[2\tau, 3\tau]$ and after the passage to the limit, implies the inequality

$$\overline{\lim_{m\to\infty}} \left| y^m(t) - \left(\mathcal{Y}(2\tau) + \int_{2\tau}^{t} f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds \right) \right| \le 2\delta F.$$

Consequently, since $\delta > 0$ is arbitrary, we have the convergence

$$y^m(t) \to \mathcal{Y}(2\tau) + \int_{2\tau}^{t} f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds, \qquad m \to \infty,$$

for $t \in [2\tau, 3\tau]$. But since

$$\mathcal{Y}(2\tau) = y_0 + \int_{0}^{\tau} f(\mathcal{Y}(s), 0)\, ds + \int_{\tau}^{2\tau} f(\mathcal{Y}(s), q\mathcal{Y}(s-\tau))\, ds$$

and $y^m(t) \to \mathcal{Y}(t)$, $m \to \infty$, it follows from the uniqueness of the limit that relation (3.6) holds on the interval $(\tau, 3\tau]$.

If $3\tau < T$, then the proof of the convergence (3.3) and (3.4) and relation (3.6) for $t \in (3\tau, T]$ can be performed successively on the intervals $(j\tau, (j+1)\tau]$ by the same scheme. The proof of the theorem is complete.

Remark. It was shown in [4] that if $0 < q < 1$, then the function $\mathcal{Y}(t)$ is a solution of the initial-value problem

$$\begin{aligned}
&\frac{d}{dt}\mathcal{Y}(t) \equiv f(\mathcal{Y}(t), q\mathcal{Y}(t-\tau)) \quad \text{for} \quad \tau < t \le T_0, \\
&\mathcal{Y}(t) \equiv y_0 + \int_{0}^{t} f(\mathcal{Y}(s), 0)\, ds \quad \text{for} \quad 0 \le t \le \tau
\end{aligned} \tag{3.20}$$

on the interval $[0, T_0]$, where

$$T_0 < \min\left\{ \frac{1-q}{L_1}, \frac{1}{L_2} \right\}.$$

It follows from Theorem 3.1 that, for each $T > \tau$, the function

$$\mathcal{Y}(t) \in C[0,T] \cap C^1(0,\tau) \cap C^1(\tau,T)$$

is a solution of the initial-value problem (3.20) on the entire interval $[0,T]$ for any $q > 0$.

4. ESTIMATES FOR THE CONVERGENCE RATE

Here we obtain estimates for the convergence rate of

$$y^m(t) \to \mathcal{Y}(t), \qquad m \to \infty,$$

and

$$y_n(t, m_0) \to y^{m_0}(t), \qquad n \to \infty$$

for given m_0.

Theorem 4.1. *The estimate*

$$\max_{[j\tau,(j+1)\tau]} |y^m(t) - \mathcal{Y}(t)| \le C_j \frac{\ln m}{m}, \qquad m \gg 1, \tag{4.1}$$

holds on any interval $[j\tau,(j+1)\tau] \subset [0,T]$, *where* $C_j > 0$ *is a constant independent of* m.

Proof. Inequality (3.9) holds on the interval $[\tau, 2\tau]$ for each $\delta \in (0,\tau)$. Then, by setting $\delta = (\ln m)/m$, we obtain the inequalities

$$\max_{[\tau,2\tau]} |y^m(t) - \mathcal{Y}(t)| \le 2\left(\frac{\ln m}{m}F + \tau L_2\left(\frac{qF}{m} + \frac{q(|y_0| + \tau F)}{m}\right)\right) e^{2\tau L_1} \le C_1 \frac{\ln m}{m}. \tag{4.2}$$

Consider the next interval $[2\tau, 3\tau]$. By taking into account the convergence (3.4) and by passing to the limit in inequality (3.19) as $l \to \infty$, we obtain

$$\max_{[2\tau,3\tau]} |y^m(t) - \mathcal{Y}(t)| \le (4\delta F + |y^m(2\tau) - \mathcal{Y}(2\tau)| + \tau L_2 \max_{[2\tau+\delta,3\tau]} |x^m(t) - q\mathcal{Y}(t-\tau)|) e^{3\tau L_1}. \tag{4.3}$$

Let us estimate $\max_{[2\tau+\delta,3\tau]} |x^m(t) - q\mathcal{Y}(t-\tau)|$. By using formulas (3.11), (3.14), and (3.15), we obtain the representation

$$x^m(t) - q\mathcal{Y}(t-\tau) = U_1^m(t) + U_2^m(t) + U_3^m(t) + X_1^m(t).$$

Consider the first term. By the definition of $U_1^m(t)$,

$$|U_1^m(t)| \le qm \int_\tau^{t-\tau} e^{-m(t-\tau-s)} |y^m(s) - \mathcal{Y}(s)|\, ds \le q \max_{[\tau,2\tau]} |y^m(\xi) - \mathcal{Y}(\xi)|.$$

This, together with inequality (4.2), implies the estimate

$$|U_1^m(t)| \le qC_1 \frac{\ln m}{m}.$$

Then, by using the estimates (3.12), (3.16), and (3.17) for arbitrary $\delta \in (0,\tau)$, we obtain the inequality

$$\max_{[2\tau,3\tau]} |y^m(t) - \mathcal{Y}(t)| \le \left(4\delta F + C_1\frac{\ln m}{m} + \tau L_2\left(qC_1\frac{\ln m}{m} + \frac{qF}{m} + 2q(|y_0| + 2\tau F)e^{-\delta m}\right)\right) e^{3\tau L_1}.$$

Consequently, just as on the preceding interval, we can set $\delta = (\ln m)/m$ and obtain the estimate (4.1) on the interval $[2\tau, 3\tau]$. The proof of inequality (4.1) for any j can be performed successively on the intervals. The proof of the theorem is complete.

Theorem 4.2. *Let m_0 be a given number. Then the inequality*

$$\max_{[0,T]} |y_n(t, m_0) - y^{m_0}(t)| \leq C n^{-1/4}, \qquad n \gg 1, \tag{4.4}$$

holds on the interval $[0, T]$ for any $T > \tau$, where $C > 0$ is a constant independent of n.

Proof. By virtue of inequalities (2.12) and (2.15), the estimate

$$|y_{n+l}(t, m_0) - y_n(t, m_0)| \leq \frac{L_1}{\lambda_1 - \lambda_2} \max_{\xi \in [0,T]} I_{n,l}(\xi, m_0) \int_0^t (\lambda_1 e^{\lambda_2 s} - \lambda_2 e^{\lambda_1 s})\, ds \tag{4.5}$$

holds on any interval $[0, T]$. It follows from the proof of Theorem 1 in [4] that

$$\max_{\xi [0,T]} I_{n,l}(\xi, m_0) \to I_n(m_0), \qquad l \to \infty; \tag{4.6}$$

moreover, the limit expression satisfies the estimate

$$I_n(m_0) \leq C_1 n^{-1/4}, \qquad n \gg 1, \tag{4.7}$$

where C_1 is a constant independent of n. By taking into account Theorem 2.1 and the convergence (4.6) and by passing to the limit as $l \to \infty$ in inequality (4.5), we obtain the estimate

$$|y^{m_0}(t) - y_n(t, m_0)| \leq C_2 I_n(m_0), \qquad t \in [0, T],$$

where $C_2 > 0$ is a constant independent of n. This, together with (4.7), implies inequality (4.4). The proof of the theorem is complete.

ACKNOWLEDGMENTS

The research was supported by the Russian Foundation for Basic Research (projects nos. 06-04-49556 and 07-01-00289), the Russian Academy of Sciences (project no. 2 in the Program for Basic Research), the Siberian Branch of the Russian Academy of Sciences (integrative projects nos. 115 and 18.13 and the integrative interdisciplinary project no. 24), and the National Scientific Foundation of the United States (FIBR PR 03-106).

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